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Convergence of Passive Scalar Fields in Ornstein–Uhlenbeck Flows to Kraichnan's Model

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We prove that the passive scalar field in the Ornstein–Uhlenbeck velocity field with wave-number dependent correlation times converges, in the white-noise limit, to that of Kraichnan's model with higher spatial regularity.

KEY WORDS: Passive scalar; white-noise limit; Kraichnan's model; stochastic flows.

1. INTRODUCTION

A passive scalar field $T(t, x)$ in a given fluid velocity $u(t, x)$ satisfies the advection-diffusion equation

$$\frac{\partial T}{\partial t} = u \cdot \nabla T + \frac{\kappa}{2} \Delta T, \qquad T(0, x) = T_0(x) \tag{1}$$

where $\kappa \geqslant 0$ is the molecular diffusivity. Kraichnan's model for passive scalar has been widely studied to understand turbulent transport in the inertial range because of its tractability (see, e.g., refs. 18, 16, 10, and the references therein). The model and its variants postulate a white-noise-in-time, *compressible or incompressible* velocity field u which can be described as the time derivative of a zero mean, isotropic Brownian field B_t with the two-time structure function

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$$\mathbb{E}[B_t(x)-B_t(y)]\otimes[B_s(x)-B_s(y)]$$

$$=\min(t,s)\int 2[1-\exp(ik\cdot(x-y))]\,a^{-1}\mathscr{E}(\eta+1,k)\,|k|^{1-d}\,dk,\qquad a>0 \tag{2}$$

Here $2a^{-1}\mathscr{E}(\eta+1,k)$ is the spatial power spectrum with

$$\mathscr{E}(\eta+1,k)=E_0(k)\,|k|^{-2\eta-1}\qquad\text{for}\quad \ell_0^{-1}\ll|k|\ll\ell_1^{-1},\quad \eta\in(0,1)$$

where $E_0(k)$ is a positive-definite matrix whose entries are homogeneous functions of degree zero, ℓ_0 and ℓ_1 are the integral and the viscous scales respectively and they determine the so-called inertial range. Below the viscous scale ℓ_1 the velocity field is smooth. The spatial Hurst exponent η characterizes the roughness of the velocity field in the inertial range and equals $2/3$ in the case of Kolmogorov's theory of turbulence. The tractability of this model lies in the Gaussian and white-noise nature of the velocity field. To fix the idea, we interpret Eq. (1) in the sense of Stratonovich's integral

$$dT=\nabla T\circ dB_t+\frac{\kappa_0}{2}\,\Delta T\,dt,\qquad \kappa_0\geqslant 0,\quad T(0,x)=T_0(x). \tag{3}$$

To study the effect of a more realistic temporal structure, one naturally considers the Ornstein–Uhlenbeck (OU) velocity field

$$u(t,x)=\frac{1}{\varepsilon}V\left(\frac{t}{\varepsilon^2},x\right) \tag{4}$$

with a similar spatial structure but a wave-number-k dependent correlation time $a^{-1}\,|k|^{-2\beta}$, $a>0$, $\beta>0$, where $\varepsilon>0$ is the scaling parameter. The two-time structure function has the spectral representation

$$\mathbb{E}[V(t,x)-V(t,y)]\otimes[V(s,x)-V(s,y)]$$

$$=\int_{\mathbb{R}^d}[1-\exp(ik\cdot(x-y))]\exp(-a\,|k|^{2\beta}\,|t-s|)\,\mathscr{E}(\alpha,k)\,|k|^{1-d}\,dk \tag{5}$$

where $\mathscr{E}$ is the power spectrum given by (6) with $1<\alpha<2$ (see refs. 8 and 9). The spatial Hurst exponent of the velocity equals $\alpha-1$ in the inertial range. The parameters α, β have the value $4/3$, $1/3$, respectively, in the case of Kolmogorov's theory of turbulence.

In this paper we study the relation between these two model. For simplicity of the presentation we set

$$\mathscr{E}(\alpha, k) = \begin{cases} E_0(k)\, |k|^{1-2\alpha}, & \text{for} \quad |k| \in (\ell_0^{-1}, \ell_1^{-1}) \\ 0, & \text{for} \quad |k| \notin (\ell_0^{-1}, \ell_1^{-1}). \end{cases} \tag{6}$$

with $\ell_0 < \infty$, $\ell_1 > 0$. We defer the discussion of the meaning of solutions of (1) and (3) until Section 2.

First we consider the situation of a non-vanishing ultraviolet cutoff $\ell_1 > 0$. We have the following correspondence principle.

Theorem 1. Let $\ell_0 < \infty$, $\ell_1 > 0$ be fixed. Let $\kappa = \kappa(\varepsilon) \geqslant 0$ and $\lim_{\varepsilon \to 0} \kappa = \kappa_0 < \infty$. Let $T_0 \in L^\infty(\mathbb{R}^d)$.

Then the solution T_t^ε of (1) with the drift (4) converges in distribution, as $\varepsilon \to 0$, in the space $D([0, t_0); L^\infty_{w^*}(\mathbb{R}^d))$, $\forall t_0 < \infty$ to the unique solution T_t of the martingale problem (cf. (20)) corresponding to Eq. (3), where the Brownian velocity field has the spatial covariance with the power spectrum $2a^{-1}\mathscr{E}(\alpha+\beta, k)$. Here $D([0, t_0); L^\infty_{w^*}(\mathbb{R}^d))$ is the space of $L^\infty(\mathbb{R}^d)$-valued right continuous processes with left limits endowed with the Skorohod metric[3] and $L^\infty_{w^*}(\mathbb{R}^d)$ is the standard space $L^\infty(\mathbb{R}^d)$ endowed with the weak* topology.

This result suggests that in the limit of rapid temporal decorrelation the OU flow resembles Kraichnan's model with a higher spatial regularity $\eta = \alpha + \beta - 1$. In particular, the strict Kolmogorov's theory $\alpha = 4/3$, $\beta = 1/3$ now corresponds to $\eta = 2/3$ in Kraichnan's model.

In the next theorem we let ℓ_1 vanish along with the scaling factor ε. In such a limit Theorem 1 is not expected to hold for *compressible flows* in the entire range of α, β for the Stratonovich correction term in the limiting Kraichnan model is well-defined only if $\alpha + \beta > 3/2$. Moreover, for $\alpha + \beta < 2$ and $\ell_1 = 0$, the Kraichnan model with *compressible* velocity field may not have a unique solution for a given initial condition due to the spatial non-Lipschitzness of the velocity field (cf. refs. 12 and 16).

Theorem 2. Suppose that the OU velocity field V is divergence-free, $\nabla \cdot V = 0$. Let $\ell_0 < \infty$ be fixed and $\ell_1 = \ell_1(\varepsilon) > 0$ such that $\lim_{\varepsilon \to 0} \ell_1 = 0$. Let $\kappa = \kappa(\varepsilon) \geqslant 0$, $\lim_{\varepsilon \to 0} \kappa = \kappa_0 < \infty$. Let $T_0 \in L^\infty \cap L^2(\mathbb{R}^d)$. If, additionally, any one of the following conditions is satisfied:

(i) $\alpha + 2\beta > 4$;

(ii) $\alpha + 2\beta = 4$, $\lim_{\varepsilon \to 0} \kappa \varepsilon^2 \sqrt{\log(1/\ell_1)} = 0$;

(iii) $3 < \alpha + 2\beta < 4$, $\lim_{\varepsilon \to 0} \kappa \varepsilon^2 \ell_1^{\alpha + 2\beta - 4} = 0$;

(iv) $\alpha+2\beta=3$, $\lim_{\varepsilon\to 0}\varepsilon\sqrt{\log(1/\ell_1)}=\lim_{\varepsilon\to 0}\kappa\varepsilon^2\ell_1^{-1}=0$;

(v) $2<\alpha+2\beta<3$, $\lim_{\varepsilon\to 0}\varepsilon\ell_1^{\alpha+2\beta-3}=\lim_{\varepsilon\to 0}\kappa\varepsilon^2\ell_1^{\alpha+2\beta-4}=0$;

(vi) $\alpha+2\beta\leqslant 2$, $\lim_{\varepsilon\to 0}\varepsilon\ell_1^{\alpha+2\beta-3}=0$,

then the convergence holds as in Theorem 1 but in the space $D([0,t_0);L^\infty_{w^*}\cap L^2_w(\mathbb{R}^d))$, $\forall t_0<\infty$ where $L^2_w(\mathbb{R}^d)$ is the usual L^2-function space endowed with the weak topology. The Brownian flow of the limiting Kraichnan's model has the the spatial power spectrum $2a^{-1}\bar{\mathscr{E}}(\alpha+\beta,k)$ where

$$\bar{\mathscr{E}}(\alpha+\beta,k)=\lim_{\ell_1\to 0}\mathscr{E}(\alpha+\beta,k).$$

Remark 1. The assumption of $L^2(\mathbb{R}^d)$-initial condition in Theorem 2 is to ensure uniqueness of the limiting Kraichnan model with $\ell_1=0$ (see Section 2). The limiting velocity field is only spatially Hölder continuous (for $\alpha+\beta<2$) with exponent $\alpha+\beta-1$.

Remark 2. For Theorem 2, when $\kappa_0>0$ and $2<\alpha+2\beta<3$, $\lim_{\varepsilon\to 0}\kappa\varepsilon^2\ell_1^{\alpha+2\beta-4}=0$ implies $\lim_{\varepsilon\to 0}\varepsilon\ell_1^{\alpha+2\beta-3}=0$.

Remark 3. For Theorem 2, in the special case of $\kappa_0=0$, the limiting Kraichnan model preserves the L^2-norm of the initial condition. On the other hand, the energy identity for the pre-limiting model (ref. 17, Chapt. III, Theorem 7.2)

$$\int |T^\varepsilon_t(x)|^2\,dx+\kappa\int_0^t\int|\nabla T^\varepsilon_t|^2\,(x)\,dx\,ds=\int|T_0(x)|^2\,dx \tag{7}$$

implies that $\|T^\varepsilon_t\|_2<\|T_0\|_2$. Consequently, the convergence in the sense of the weak-L^2 topology in Theorem 2 implies that $\lim_{\varepsilon\to 0}\|T^\varepsilon_t\|_2=\|T_t\|_2$ and that the convergence is indeed in the strong L^2 sense.

Finally we note that the Gaussianity of the velocity field is not essential to the results. It has been used in the proofs to control the first 4 moments of the velocity fields and to have a mild decay in the tail distributions of the velocity fields (cf. (36)). For non-Gaussian velocity fields the *temporal* decorrelation needs to be sufficiently fast such that a certain integral is well-defined. For instance, processes having an absolutely integrable uniform-mixing coefficient would suffice (see (29) and the remark thereafter). The comparable result in ref. 13 requires a faster-than-Gaussian decay in the tail distributions and does not apply here. It also requires spatial regularity in the velocity fields (see also ref. 4).

2. FORMULATION OF SOLUTIONS

From the general theory of parabolic partial differential equations,[11] for any fixed $\kappa > 0$, $\varepsilon > 0$, the solution $T_t^\varepsilon(x)$ is a $C^{2+\eta}$-function, with any $0 < \eta < \alpha - 1$. But the solutions T_t^ε may lose all the regularity as $\kappa \to 0$, $\varepsilon \to 0$. So we consider the weak formulation of the equation:

$$\langle T^\varepsilon, \theta\rangle - \langle T_0, \theta\rangle = \frac{\kappa}{2}\int_0^t \langle T_s^\varepsilon, \Delta\theta\rangle\, ds - \frac{1}{\varepsilon}\int_0^t \left\langle T_s^\varepsilon, \nabla\cdot\left(\theta V\left(\frac{s}{\varepsilon^2}, \cdot\right)\right)\right\rangle ds \quad (8)$$

for any test function $\theta \in C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact supports. We view T_t^ε as distribution-valued processes. The solutions T_t^ε can be represented as

$$T_t^\varepsilon(x) = \mathbb{M}[T_0(\Phi_0^{t,\varepsilon}(x))] \quad (9)$$

where $\Phi_s^{t,\varepsilon}$ is the unique stochastic flow of the SDE

$$d\Phi_s^{t,\varepsilon}(x) = -\frac{1}{\varepsilon} V\left(\Phi_s^{t,\varepsilon}(x), \frac{s}{\varepsilon^2}\right) ds + \kappa^{1/2}\, dw(t), \qquad 0 \leqslant s \leqslant t \quad (10)$$

$$\Phi_t^{t,\varepsilon}(x) = x. \quad (11)$$

In view of the averaging in the representation (9) we have

Proposition 1.

$$\|T_t^\varepsilon\|_\infty \leqslant \|T_0\|_\infty \quad \text{a.s.}$$

One also has that

$$\mathbb{E}\{\|T_t^\varepsilon\|_p^p\} \leqslant \|T_0\|_p^p, \qquad \forall p \geqslant 1. \quad (12)$$

Indeed, by the spatial homogeneity of the field V, the distribution of $\Phi_s^{t,\varepsilon}(x)$ is the same as the distribution of $\Phi_s^{t,\varepsilon}(0) + x$ for each fixed x. Hence we have

$$\mathbb{E}[\|T_t^\varepsilon\|_p^p] \leqslant \int \mathbb{M}\mathbb{E}[T_0^p(\Phi_0^{t,\varepsilon}(x))]\, dx = \mathbb{M}\mathbb{E}\left[\int T_0^p(\Phi_0^{t,\varepsilon}(0) + x)\, dx\right] = \|T_0\|_p^p.$$

Proposition 1 (resp. (12)) says that, for $T_0 \in L^\infty$(resp. L^p), T_t^ε is almost surely a L^∞ (resp. L^p)-function for every $t \geqslant 0$.

For tightness as well as identification of the limit, the following infinitesimal operator $\mathscr{A}^\varepsilon$ will play an important role. Let $V_t^\varepsilon \equiv V(t/\varepsilon^2, \cdot)$. Let $\mathscr{F}_t^\varepsilon$ be the σ-algebras generated by $\{V_s^\varepsilon, s \leqslant t\}$ and $\mathbb{E}_t^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathscr{F}_t^\varepsilon$. Let $\mathscr{M}^\varepsilon$ be the space of measurable function adapted to $\{\mathscr{F}_t^\varepsilon, \forall t\}$ such that $\sup_{t<t_0} \mathbb{E}\,|f(t)| < \infty$. We say $f(\cdot) \in \mathscr{D}(\mathscr{A}^\varepsilon)$, the domain of $\mathscr{A}^\varepsilon$, and $\mathscr{A}^\varepsilon f = g$ if $f, g \in \mathscr{M}^\varepsilon$ and for $f^\delta(t) \equiv \delta^{-1}[\mathbb{E}_t^\varepsilon f(t+\delta) - f(t)]$ we have

$$\sup_{t,\delta} \mathbb{E}\,|f^\delta(t)| < \infty$$

$$\lim_{\delta \to 0} \mathbb{E}\,|f^\delta(t) - g(t)| = 0, \qquad \forall t.$$

For $f(t) = \phi(\langle T_t^\varepsilon, \theta \rangle)$, $f'(t) = \phi'(\langle T_t^\varepsilon, \theta \rangle)$, $\forall \phi \in C_c^3(\mathbb{R})$ (i.e., C^3-function with a compact support) we have the following expression from (8) and the chain rule

$$\mathscr{A}^\varepsilon f(t) = \frac{\kappa}{2} f'(t) \langle T_t^\varepsilon, \Delta\theta \rangle - \frac{1}{\varepsilon} f'(t) \langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta) \rangle \tag{13}$$

where

$$\mathscr{V}_t^\varepsilon(\theta) \equiv \nabla \cdot [\theta V_t^\varepsilon]. \tag{14}$$

A main property of $\mathscr{A}^\varepsilon$ is that

$$f(t) - \int_0^t \mathscr{A}^\varepsilon f(s)\, ds \quad \text{is a } \mathscr{F}_t^\varepsilon\text{-martingale}, \qquad f \in \mathscr{D}(\mathscr{A}^\varepsilon). \tag{15}$$

Also,

$$\mathbb{E}_t^\varepsilon f(s) - f(t) = \int_t^s \mathbb{E}_t^\varepsilon \mathscr{A}^\varepsilon f(\tau)\, d\tau \qquad \forall s > t \quad \text{a.s.} \tag{16}$$

(see ref. 14). We can view T_t^ε as the distribution-valued stochastic solutions to the martingale problem (15).

Likewise we formulate the solutions for the Kraichnan's model (3) as the solutions to the corresponding martingale problem. We will first describe the limiting martingale problem for Theorem 1 and then discuss the changes due to $\ell_1 \to 0$ in Theorem 2. We rewrite (3) as an Itô's SDE

$$dT_t = \left(\frac{\kappa_0}{2}\,\Delta + \frac{1}{a}\,\mathscr{B}\right) T_t\, dt + \sqrt{2}\, a^{-1/2}\, \nabla T_t \cdot dW_t^{(1)} \tag{17}$$

where $W_t^{(1)}(x)$ is the Brownian vector field with the spatial covariance

$$\Gamma^{(1)}(x-y) = \int \exp(ik\cdot(x-y))\, \mathscr{E}(\alpha+\beta, k)\, |k|^{1-d}\, dk$$

and the operator $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is given by

$$\mathscr{B}_1 \phi = \sum_i \left[\frac{\partial}{\partial x_i}\, \Gamma_{ij}^{(1)}(0)\right] \frac{\partial \phi}{\partial x_j} \tag{18}$$

$$\mathscr{B}_2 \phi = \sum_{i,j} \Gamma_{ij}^{(1)}(0)\, \frac{\partial^2 \phi}{\partial x_i\, \partial x_j}. \tag{19}$$

Equation (17) can be formulated as the martingale problem: Find a measure $\mathbb{P}$ (of T_t) on the subspace of $D([0, t_0); L^\infty_{w*}(\mathbb{R}^d))$ whose elements have a given initial data in $L^\infty_{w*}(\mathbb{R}^d)$ such that

$$\begin{aligned} f(\langle T_t, \theta\rangle) - \int_0^t \Big\{ f'(\langle T_t, \theta\rangle) \Big[\frac{\kappa_0}{2}\,\langle T_s, \Delta\theta\rangle + \frac{1}{a}\,\langle T_s, \mathscr{B}^*\theta\rangle\Big] \\ + \frac{1}{a}\, f''(\langle T_t, \theta\rangle)\langle \theta, \mathscr{K}_{T_s}^{(1)}\theta\rangle \Big\}\, ds \end{aligned}$$

is a martingale w.r.t. the filtration of a cylindrical Wiener process,

for each $f \in C_c^3(\mathbb{R})$ (20)

where $\mathscr{B}^*$ is the adjoint of $\mathscr{B}$ and $\mathscr{K}_{T_t}^{(1)}$ is a positive-definite operator given formally as

$$\mathscr{K}_{T_t}^{(1)}\theta = \int \theta(y)\, \nabla T_t(x) \cdot \Gamma^{(1)}(x-y)\, \nabla T_t(y)\, dy \tag{21}$$

such that

$$\langle \theta_1, \mathscr{K}_\phi^{(1)} \theta_2\rangle = \iint \phi(x)\, \phi(y)\, G_{\theta_1,\theta_2}^{(1)}(x, y)\, dx\, dy \tag{22}$$

$$G_{\theta_1,\theta_2}^{(1)} \equiv \sum_{i,j} \frac{\partial^2}{\partial x_i\, \partial y_j}\, [\theta_1(x)\, \theta_2(y)\, \Gamma_{ij}^{(1)}(x-y)]. \tag{23}$$

When $\ell_1 \to 0$ (Theorem 2) $\Gamma^{(1)}$ in the preceding discussion should be replaced by

$$\bar{\Gamma}^{(1)}(x-y) = \lim_{\ell_1 \to 0} \Gamma^{(1)}(x-y) \tag{24}$$

and all objects (such as $\mathscr{B}$, G_{θ_1,θ_2}, $\mathscr{K}^{(1)}_{T_t}$) related to $\Gamma^{(1)}$ should be replaced accordingly (by $\bar{\mathscr{B}}$, $\bar{G}_{\theta_1,\theta_2}$, $\bar{\mathscr{K}}^{(1)}_{T_t}$). In particular, $\bar{\mathscr{B}}_1$ is well-defined only for $\alpha+\beta > 3/2$ in general for compressible flows. Namely, the martingale problem (20) is not well defined in the compressible case unless the limiting Brownian velocity has a spatial Hurst exponent which is bigger than $1/2$.

In the case of divergence-free vector fields, $\bar{\mathscr{B}}_1 = 0$ and

$$\bar{\mathscr{B}}\phi = \sum_{i,j} \bar{\Gamma}^{(1)}_{ij}(0) \frac{\partial^2 \phi}{\partial x_i \, \partial x_j}. \tag{25}$$

Also,

$$\bar{G}^{(1)}_{\theta_1,\theta_2} \equiv \sum_{i,j} \bar{\Gamma}^{(1)}_{ij}(x-y) \frac{\partial \theta_1(x)}{\partial x_i} \frac{\partial \theta_2(y)}{\partial y_j}.$$

2.1. Uniqueness of the Limiting Kraichnan Model

When $\lim_{\varepsilon\to 0} \ell_1 > 0$ the limit Brownian velocity field is spatially smooth and generates a unique stochastic flow on $\mathbb{R}^d$ (refs. 1 and 2) from which it follows the uniqueness of the martingale solution.

When $\lim_{\varepsilon\to 0} \ell_1 = 0$ the limiting velocity field is only spatially Hölder continuous and we establish the uniqueness of the martingale solution by proving the uniqueness of the n-point correlation function

$$F^t_n(x_1, x_2, x_3,..., x_n) \equiv \mathbb{E}_{T_0}[T_t(x_1)\, T_t(x_2) \cdots T_t(x_n)].$$

The evolution of the n-point correlation function is given by a weakly continuous (hence strongly continuous) sub-Markovian semigroup on $L^p(\mathbb{R}^{nd})$, $\forall p \in (1, \infty)$ whose generator $\mathscr{L}_n$ can be deduced by taking the test function $f(r) = r^n$ in the martingale formulation, i.e.,

$$\begin{aligned} &\mathscr{L}_n \Phi(x_1,..., x_n) \\ &\quad \equiv \frac{\kappa_0}{2} \sum_{j=1}^n \Delta_{x_j} \Phi + \frac{1}{a} \sum_{i,j=1}^n \bar{\Gamma}^{(1)}(x_i - x_j) : \nabla_{x_i} \nabla_{x_j} \Phi, \qquad \Phi \in C^\infty_c(\mathbb{R}^{nd}), \quad \kappa_0 \geqslant 0. \end{aligned} \tag{26}$$

Note that the symmetric operator $\mathscr{L}_n$ (26) is an essentially self-adjoint positive operator on $C_c^\infty(\mathbb{R}^{nd})$, which then induces a *unique* symmetric Markov semigroup of contractions on $L^2(\mathbb{R}^{nd})$. The essential self-adjointness is due to the sub-Lipschitz growth of the square-root of $\bar{\Gamma}^{(1)}(x_1-x_2)$ at large $|x_1|$, $|x_2|$ (hence no escape to infinity).[6]

3. PROOF OF THEOREM 1

The proofs are a refinement of that of ref. 5 to deal with the wave-number dependence of the correlation time and the lack of spatial regularity in the velocity fields. For the reader's convenience, we will repeat some of the calculations in ref. 5 and refer the reader to ref. 15 for the full exposition of the perturbed test function method used here. The perturbed test function method is initiated in ref. 19.

In the sequel we will adopt the following notation

$$f(t) \equiv f(\langle T_t^\varepsilon, \theta\rangle), \qquad f'(t) \equiv f'(\langle T_t^\varepsilon, \theta\rangle), \qquad f''(t) \equiv f''(\langle T_t^\varepsilon, \theta\rangle),$$

$$f'''(t) \equiv f'''(\langle T_t^\varepsilon, \theta\rangle) \qquad \forall f \in C_c^3(\mathbb{R}).$$

Namely, the prime stands for the differentiation w.r.t. the original argument ($\langle T_t^\varepsilon, \theta\rangle$ not t) of f, f', etc.

3.1. Tightness

A family of distribution-valued right-continuous with left limits processes $\{T^\varepsilon, 0<\varepsilon<1\}$ is tight if and only if the family of real-valued, right-continuous with left limits processes $\{\langle T^\varepsilon, \theta\rangle, 0<\varepsilon<1\}$ is tight for all $\theta \in C_c^\infty(\mathbb{R}^d)$. We use the tightness criterion of ref. 15 (Chap. 3, Theorem 4), namely, we will prove: Firstly,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\{\sup_{t<t_0} |\langle T^\varepsilon, \theta\rangle| \geqslant N\} = 0, \qquad \forall t_0 < \infty. \tag{27}$$

Secondly, for each $f \in C_c^3(\mathbb{R})$ there is a sequence $f^\varepsilon(t) \in \mathscr{D}(\mathscr{A}^\varepsilon)$ such that for each $t_0<\infty$ $\{\mathscr{A}^\varepsilon f^\varepsilon(t), 0<\varepsilon<1, 0<t<t_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{t<t_0} |f^\varepsilon(t) - f(\langle T^\varepsilon, \theta\rangle)| \geqslant \delta\} = 0, \qquad \forall \delta > 0. \tag{28}$$

Then it follows that the laws of $\{\langle T^\varepsilon, \theta\rangle, 0<\varepsilon<1\}$ are tight in the space $D([0, t_0); \mathbb{R})$.

Condition (27) is satisfied as a result of Proposition 1. Let

$$f_1^\varepsilon(t) \equiv \frac{1}{\varepsilon} \int_t^\infty \mathbb{E}_t^\varepsilon f'(t) \langle T_t^\varepsilon, \mathscr{V}_s^\varepsilon(\theta) \rangle \, ds \tag{29}$$

be the 1-st perturbation of $f(t)$. Equation (29) is well-defined in view of the spectral representation for the Gaussian process

$$\mathbb{E}_t^\varepsilon V_s^\varepsilon = \int e^{ix \cdot k} e^{-a |k|^{2\beta} |s-t| \varepsilon^{-2}} \hat{V}_t^\varepsilon(dk), \qquad \forall s \geqslant t. \tag{30}$$

For non-Gaussian processes, absolutely integrable uniform-mixing coefficients would suffice to define f_1^ε.[7] We then have

$$f_1^\varepsilon(t) = \frac{\varepsilon}{a} f'(t) \langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta) \rangle \tag{31}$$

with

$$\tilde{\mathscr{V}}_t^\varepsilon(\theta) = \nabla \cdot [\theta \tilde{V}_t^\varepsilon] \tag{32}$$

$$\tilde{V}_t^\varepsilon \equiv \tilde{V}\left(\frac{t}{\varepsilon^2}, \cdot\right) \equiv \varepsilon^{-2} \int_t^\infty \mathbb{E}_t^\varepsilon V_s^\varepsilon \, ds \tag{33}$$

where $\tilde{V}$ has the power spectrum $\mathscr{E}(\alpha + 2\beta, k)$.

Proposition 2.

$$\lim_{\varepsilon \to 0} \sup_{t < t_0} \mathbb{E} |f_1^\varepsilon(t)| = 0, \qquad \lim_{\varepsilon \to 0} \sup_{t < t_0} |f_1^\varepsilon(t)| = 0 \quad \text{in probability.}$$

Proof. By Proposition 1 we have

$$\mathbb{E}[|f_1^\varepsilon(t)|] \leqslant \frac{\varepsilon}{a} \|f'\|_\infty \|T_0\|_\infty \left[\|\theta\|_\infty \int_{|x| \leqslant M} \mathbb{E} |\tilde{V}_t^\varepsilon(x)| \, dx + \|\nabla\theta\|_\infty \int_{|x| \leqslant M} \mathbb{E} |\nabla \cdot \tilde{V}_t^\varepsilon| \, dx \right] \tag{34}$$

and

$$\sup_{t < t_0} |f_1^\varepsilon(t)| \leqslant \frac{\varepsilon}{a} \|f'\|_\infty \|T_0\|_\infty \times \left[\|\theta\|_\infty \sup_{t < t_0} \int_{|x| \leqslant M} |\tilde{V}_t^\varepsilon(x)| \, dx + \|\nabla\theta\|_\infty \sup_{t < t_0} \int_{|x| \leqslant M} |\nabla \cdot \tilde{V}_t^\varepsilon| \, dx \right]. \tag{35}$$

By the temporal stationarity of $\tilde{V}_t^\varepsilon$ we can replace $\mathbb{E}\,|\tilde{V}_t^\varepsilon(x)|$, $\mathbb{E}\,|\nabla\cdot\tilde{V}_t^\varepsilon(x)|$ in (34) by $\mathbb{E}\,|\tilde{V}(0,x)|$, $\mathbb{E}\,|\nabla\cdot\tilde{V}(0,x)|$. By the Gaussianity, temporal stationarity and spatial homogeneity of $\tilde{V}$, we can replace $\sup_{t<t_0}\int_{|x|\leqslant M}|\tilde{V}_t^\varepsilon(x)|\,dx$ in (35) by

$$M^d \sup_{\substack{|x|\leqslant M\\ t\leqslant t_0}}\left|\tilde{V}\left(\frac{t}{\varepsilon},x\right)\right| \leqslant C\log\left[\frac{M^d t_0}{\varepsilon^2}\right] = o\left(\frac{1}{\varepsilon}\right) \tag{36}$$

with a random constant C possessing a distribution with a finite moment (indeed, a Gaussian-like tail by Chernoff's bound). A similar inequality holds for $\nabla\cdot\tilde{V}$. Proposition 2 now follows from (34)–(36). ▮

Set $f^\varepsilon(t)=f(t)-f_1^\varepsilon(t)$. A straightforward calculation yields

$$\begin{aligned}\mathscr{A}^\varepsilon f_1^\varepsilon = & -\frac{\kappa\varepsilon}{2a}f''(t)\langle T_t^\varepsilon, \Delta\theta\rangle\langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle + \frac{\kappa\varepsilon}{2a}f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle\\ & + \frac{1}{a}f''(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle - \frac{1}{a}f'(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle\\ & + \frac{1}{\varepsilon}f'(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta)\rangle\end{aligned}$$

and, hence

$$\begin{aligned}\mathscr{A}^\varepsilon f^\varepsilon(t) = & \frac{\kappa}{2}f'(t)\langle T_t^\varepsilon, \Delta\theta\rangle - \frac{1}{a}f'(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle\\ & -\frac{1}{a}f''(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle\\ & +\frac{\kappa\varepsilon}{2a}[f''(t)\langle T_t^\varepsilon, \Delta\theta\rangle\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta)\rangle - f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle]\\ = & A_1^\varepsilon(t)+A_2^\varepsilon(t)+A_3^\varepsilon(t)+A_4^\varepsilon(t)\end{aligned} \tag{37}$$

where $A_2^\varepsilon(t)$ and $A_3^\varepsilon(t)$ are the $O(1)$ statistical coupling terms.

For the tightness criterion stated in the beginnings of the section, it remains to show

Proposition 3. $\{\mathscr{A}^\varepsilon f^\varepsilon\}$ are uniformly integrable and

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|A_4^\varepsilon(t)|=0.$$

Proof. We show that $\{A_i^\varepsilon\}$, $i=1,2,3,4$ are uniformly integrable. To see this, we have the following estimates.

$$|A_1^\varepsilon(t)| = \frac{\kappa}{2}\,|f'(t)\langle T_t^\varepsilon, \Delta\theta\rangle| \leqslant \frac{\kappa}{2}\,\|f'\|_\infty\,\|T_0\|_\infty\,\|\Delta\theta\|_1$$

Thus A_1^ε is uniformly integrable since it is uniformly bounded.

$$\begin{aligned}|A_2^\varepsilon(t)| &= \frac{1}{a}\,|f'(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle| \\ &\leqslant \frac{C}{a}\,\|f'\|_\infty\,\|T_0\|_\infty\left[\int_{|x|<M}|V_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\cdot V_t^\varepsilon|^2\,dx\right]^{1/2} \\ &\quad\times\left[\int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\cdot\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\nabla\cdot\tilde{V}_t^\varepsilon|^2\,dx\right]^{1/2}.\end{aligned}$$

Thus A_2^ε is uniformly integrable in view of the uniform boundedness of the 4th moment of V_t^ε, $\tilde{V}_t^\varepsilon$, and their spatial derivatives due to Gaussianity and the ultraviolet cutoff $\ell_1 > 0$.

$$\begin{aligned}|A_3^\varepsilon(t)| &= \frac{1}{a}\,|f''(t)\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle| \\ &\leqslant \frac{C}{a}\,\|f'\|_\infty\,\|T_0\|_\infty^2\left[\int_{|x|<M}|V_t^\varepsilon|^2\,dx\right. \\ &\quad\left. + \int_{|x|<M}|\nabla\cdot V_t^\varepsilon|^2\,dx + \int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\cdot\tilde{V}_t^\varepsilon|^2\,dx\right].\end{aligned}$$

Thus A_3^ε is uniformly integrable for the similar reason that A_2^ε is uniformly integrable.

$$\begin{aligned}|A_4^\varepsilon| &= \frac{\kappa\varepsilon}{2a}\,|f''(t)\langle T_t^\varepsilon, \Delta\theta\rangle\langle T_t^\varepsilon, \tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle - f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle| \\ &\leqslant \frac{C\kappa\varepsilon}{2a}\left[\|f''\|_\infty\,\|T_0\|_\infty^2\left[\int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\cdot\tilde{V}_t^\varepsilon|^2\,dx\right]^{1/2} + \|f'\|_\infty\,\|T_0\|_\infty\right. \\ &\quad\times\left[\int_{|x|<M}|\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla\tilde{V}_t^\varepsilon|^2\,dx + \int_{|x|<M}|\nabla^2\tilde{V}_t^\varepsilon|^2\,dx\right. \\ &\quad\left.\left. + \int_{|x|<M}|\nabla^2\nabla\cdot\tilde{V}_t^\varepsilon|^2\,dx\right]^{1/2}\right].\end{aligned} \tag{38}$$

Due to the fixed cutoff $\ell_1 > 0$, the higher derivatives of $\tilde{V}_t^\varepsilon$ do not cause any difficulty and they all have uniformly bounded, say, the 4th moments. Hence A_4^ε is uniformly integrable. Clearly

$$\lim_{\varepsilon \to 0} \sup_{t < t_0} \mathbb{E}\,|A_4^\varepsilon(t)| = 0. \quad \blacksquare$$

3.2. Identification of the Limit

Once the tightness is established we can use another result in ref. 15 (Chapter 3, Theorem 2) to identify the limit. Let $\bar{\mathscr{A}}$ be a diffusion or jump diffusion operator such that there is a unique solution ω_t in the subspace of $D([0, t_0); L^\infty_{w*}(\mathbb{R}^d))$, $\forall t_0 < \infty$, whose elements have the given initial data in $L^\infty_{w*}(\mathbb{R}^d)$ and that the solution satisfies the martingale property, i.e.,

$$f(\omega_t) - \int_0^t \bar{\mathscr{A}} f(\omega_s)\, ds \tag{39}$$

is a martingale. We shall show that for each $f \in C_c^3(\mathbb{R})$ there exists $f^\varepsilon \in \mathscr{D}(\mathscr{A}^\varepsilon)$ such that

$$\sup_{t < t_0, \varepsilon} \mathbb{E}\,|f^\varepsilon(t) - f(\langle T_t^\varepsilon, \theta\rangle)| < \infty \tag{40}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\,|f^\varepsilon(t) - f(\langle T_t^\varepsilon, \theta\rangle)| = 0, \qquad \forall t < t_0 \tag{41}$$

$$\sup_{t < t_0, \varepsilon} \mathbb{E}\,|\mathscr{A}^\varepsilon f^\varepsilon(t) - \bar{\mathscr{A}} f(\langle T_t^\varepsilon, \theta\rangle)| < \infty \tag{42}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\,|\mathscr{A}^\varepsilon f^\varepsilon(t) - \bar{\mathscr{A}} f(\langle T_t^\varepsilon, \theta\rangle)| = 0, \qquad \forall t < t_0. \tag{43}$$

Then the aforementioned theorem implies that any tight processes $\langle T_t^\varepsilon, \theta\rangle$ converges in law to the unique process generated by $\bar{\mathscr{A}}$. As before we adopt the notation $f(t) = f(\langle T_t^\varepsilon, \theta\rangle)$.

For this purpose, we introduce the next perturbations $f_2^\varepsilon, f_3^\varepsilon$. Let

$$A_2^{(1)}(\phi) \equiv \langle \theta, \mathscr{K}_\phi^{(1)} \theta\rangle \tag{44}$$

$$A_3^{(1)}(\phi) \equiv \langle \phi, \mathbb{E}[\mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))]\rangle \tag{45}$$

where the positive-definite operator $\mathscr{K}^{(1)}_{\phi}$ is defined in (21). It is easy to see that

$$A^{(1)}_2(\phi) = \mathbb{E}[\langle \phi, \mathscr{V}^{\varepsilon}_t(\theta)\rangle\langle \phi, \tilde{\mathscr{V}}^{\varepsilon}_t(\theta)\rangle] \tag{46}$$

$$A^{(1)}_3(\phi) = \langle \mathscr{B}\phi, \theta\rangle \tag{47}$$

where the operator $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is given by (18) and (19).

Define

$$f^{\varepsilon}_2(t) \equiv \frac{1}{a} f''(t) \int_t^{\infty} \mathbb{E}^{\varepsilon}_t[\langle T^{\varepsilon}_t, \mathscr{V}^{\varepsilon}_s(\theta)\rangle\langle T^{\varepsilon}_t, \tilde{\mathscr{V}}^{\varepsilon}_s(\theta)\rangle - A^{(1)}_2(T^{\varepsilon}_t)]\, ds$$

$$f^{\varepsilon}_3(t) \equiv \frac{1}{a} f'(t) \int_t^{\infty} \mathbb{E}^{\varepsilon}_t[\langle T^{\varepsilon}_t, \mathscr{V}^{\varepsilon}_s(\tilde{\mathscr{V}}^{\varepsilon}_s(\theta))\rangle - A^{(1)}_3(T^{\varepsilon}_t)]\, ds.$$

Let

$$G^{(2)}_{\theta_1, \theta_2} \equiv \sum_{i,j} \frac{\partial^2}{\partial x_i\, \partial y_j} [\theta_1(x)\, \theta_2(y)\, \Gamma^{(2)}_{ij}(x-y)]$$

$$\langle \theta_1, \mathscr{K}^{(2)}_{\phi}\theta_2\rangle \equiv \iint \phi(x)\, \phi(y)\, G^{(2)}_{\theta_1,\theta_2}(x, y)\, dx\, dy$$

where the covariance function $\Gamma^{(2)}(x-y) \equiv \mathbb{E}[\tilde{V}^{\varepsilon}_t(x) \otimes \tilde{V}^{\varepsilon}_t(y)]$ has the spectral density $\mathscr{E}(\alpha + 2\beta, k)$, and let

$$A^{(2)}_2(\phi) \equiv \langle \theta, \mathscr{K}^{(2)}_{\phi}\theta\rangle$$

$$A^{(2)}_3(\phi) \equiv \langle \phi, \mathbb{E}[\tilde{\mathscr{V}}^{\varepsilon}_t(\tilde{\mathscr{V}}^{\varepsilon}_t(\theta))]\rangle.$$

We then have

$$f^{\varepsilon}_2(t) = \frac{\varepsilon^2}{2a^2} f''(t)[\langle T^{\varepsilon}_t, \tilde{\mathscr{V}}^{\varepsilon}_t(\theta)\rangle^2 - A^{(2)}_2(T^{\varepsilon}_t)] \tag{48}$$

and similarly

$$f^{\varepsilon}_3(t) = \frac{\varepsilon^2}{2a^2} f'(t)[\langle T^{\varepsilon}_t, \tilde{\mathscr{V}}^{\varepsilon}_t(\tilde{\mathscr{V}}^{\varepsilon}_t(\theta))\rangle - A^{(2)}_3(T^{\varepsilon}_t)]. \tag{49}$$

In view of the pre-factor ε in (48) and (49) and the fact that all terms involved are regular and uniformly bounded, we have

Proposition 4.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|f_2^\varepsilon(t)|=0,\qquad \lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|f_3^\varepsilon(t)|=0.$$

The proof of Proposition 4 is analogous to that of Proposition 2. We have

$$\mathscr{A}^\varepsilon f_2^\varepsilon(t)=\frac{1}{a}\,f''(t)[\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\theta)\rangle\langle T_t^\varepsilon,\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle-A_2^{(1)}(T_t^\varepsilon)]+R_2^\varepsilon(t)$$

$$\mathscr{A}^\varepsilon f_3^\varepsilon(t)=\frac{1}{a}\,f'(t)[\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle-A_3^{(1)}(T_t^\varepsilon)]+R_3^\varepsilon(t)$$

with

$$\begin{aligned}R_2^\varepsilon(t)=&\;f'''(t)\left[\frac{\varepsilon^2\kappa}{4a^2}\langle T_t^\varepsilon,\Delta\theta\rangle-\frac{\varepsilon}{2a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\theta)\rangle\right][\langle T_t^\varepsilon,\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle^2-A_2^{(2)}(T_t^\varepsilon)]\\ &+f''(t)\langle T_t^\varepsilon,\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle\left[\frac{\kappa\varepsilon^2}{2a^2}\langle T_t^\varepsilon,\Delta\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle-\frac{\varepsilon}{a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle\right]\\ &-f''(t)\left[\frac{\kappa\varepsilon^2}{4a^2}\langle T_t^\varepsilon,\Delta G_\theta^{(2)}T_t^\varepsilon\rangle-\frac{\varepsilon}{a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(G_\theta^{(2)}T_t^\varepsilon)\rangle\right]\end{aligned}\tag{50}$$

where in (50) $G_\theta^{(2)}$ denotes the operator

$$G_\theta^{(2)}\phi\equiv\int G_{\theta,\theta}^{(2)}(x,y)\,\phi(y)\,dy,$$

and similarly

$$\begin{aligned}R_3^\varepsilon(t)=&\;f''(t)\left[\frac{\kappa\varepsilon^2}{4a^2}\langle T_t^\varepsilon,\Delta\theta\rangle-\frac{\varepsilon}{2a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\theta)\rangle\right]\\ &\times[\langle T_t^\varepsilon,\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle-A_3^{(2)}(T_t^\varepsilon)]\\ &+f'(t)\left[\frac{\kappa\varepsilon^2}{4a^2}\langle T_t^\varepsilon,\Delta\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))\rangle-\frac{\varepsilon}{2a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta)))\rangle\right]\\ &-f'(t)\left[\frac{\kappa\varepsilon^2}{4a^2}\langle T_t^\varepsilon,\Delta\mathbb{E}[\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))]\rangle\right.\\ &\left.+\frac{\varepsilon}{2a^2}\langle T_t^\varepsilon,\mathscr{V}_t^\varepsilon(\mathbb{E}[\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta))])\rangle\right].\end{aligned}$$

Now all terms appearing in $R_2^\varepsilon(t)$ and $R_3^\varepsilon(t)$ are regular and uniformly bounded, we easily have

Proposition 5.

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|R_2^\varepsilon(t)|=0,\qquad \lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|R_3^\varepsilon(t)|=0.$$

Set

$$R^\varepsilon(t)=A_4^\varepsilon(t)+R_2^\varepsilon(t)+R_3^\varepsilon(t).$$

It follows from Propositions 3 and 5 that

$$\lim_{\varepsilon\to 0}\sup_{t<t_0}\mathbb{E}\,|R^\varepsilon(t)|=0.$$

Recall that

$$\begin{aligned}M_t^\varepsilon(\theta)&=f^\varepsilon(t)-\int_0^t \mathscr{A}^\varepsilon f^\varepsilon(s)\,ds\\&=f(t)-f_1^\varepsilon(t)+f_2^\varepsilon(t)+f_3^\varepsilon(t)-\int_0^t\frac{\kappa}{2}f'(t)\langle T_t^\varepsilon,\Delta\theta\rangle\,ds\\&\quad-\int_0^t\frac{1}{a}\left[f''(t)(\langle T_s^\varepsilon,\theta\rangle)\,A_2^{(1)}(T_s^\varepsilon)+f'(t)\,A_3^{(1)}(T_s^\varepsilon)\right]ds-\int_0^t R^\varepsilon(s)\,ds\end{aligned}$$

is a martingale. Now that (40)–(43) are satisfied we can identify the limiting martingale to be

$$M_t(\theta)=f(t)-\int_0^t\left\{f'(s)\left[\frac{\kappa_0}{2}\langle T_s,\Delta\theta\rangle+\frac{1}{a}A_3^{(1)}(T_s)\right]+\frac{1}{a}f''(s)\,A_2^{(1)}(T_s)\right\}ds. \tag{51}$$

Since $\langle T_t^\varepsilon,\theta\rangle$ is uniformly bounded

$$|\langle T_t^\varepsilon,\theta\rangle|\leqslant \|T_0\|_\infty\,\|\theta\|_1$$

we have the convergence of the second moment

$$\lim_{\varepsilon\to 0}\mathbb{E}\{\langle T_t^\varepsilon,\theta\rangle^2\}=\mathbb{E}\{\langle T_t,\theta\rangle^2\}.$$

Use $f(r)=r$ and r^2 in (51)

$$M_t^{(1)}(\theta)=\langle T_t,\theta\rangle-\int_0^t\left[\frac{\kappa_0}{2}\langle T_s,\Delta\theta\rangle+\frac{1}{a}A_3^{(1)}(T_s)\right]ds$$

is a martingale with the quadratic variation

$$[M^{(1)}(\theta),M^{(1)}(\theta)]_t=\frac{2}{a}\int_0^t A_2^{(1)}(T_s)\,ds=\frac{2}{a}\int_0^t\langle\theta,\mathscr{K}_{T_s}^{(1)}\theta\rangle\,ds.$$

Therefore,

$$M_t^{(1)}=\sqrt{\frac{2}{a}}\int_0^t\sqrt{\mathscr{K}_{T_s}^{(1)}}\,dW_s$$

where W_s is a cylindrical Wiener process (i.e., $dW_t(x)$ is a space-time white noise field) and $\sqrt{\mathscr{K}_{T_s}^{(1)}}$ is the square-root of the positive-definite operator given in (21). From (44) and (47) we see that the limiting process T_t is the (assumed unique) distributional solution to the martingale problem (20) of the Itô's equation

$$\begin{aligned}dT_t&=\left(\frac{\kappa_0}{2}\Delta+\frac{1}{a}\mathscr{B}\right)T_t\,dt+\sqrt{2a^{-1}\mathscr{K}_{T_t}^{(1)}}\,dW_t\\&=\left(\frac{\kappa_0}{2}\Delta+\frac{1}{a}\mathscr{B}\right)T_t\,dt+\sqrt{2}\,a^{-1/2}\,\nabla T_t\cdot dW_t^{(1)}\end{aligned}$$

where the operator $\mathscr{B}=\mathscr{B}_1+\mathscr{B}_2$ is given by (18) and (19) and $W_t^{(1)}$ is the Brownian vector field with the spatial covariance $\Gamma^{(1)}(x-y)$.

4. PROOF OF THEOREM 2

The argument is the same as before except with

$$\mathscr{V}_t^{\varepsilon}(\theta)\equiv V_t^{\varepsilon}\cdot\nabla\theta$$

$$\widetilde{\mathscr{V}}_t^{\varepsilon}(\theta)\equiv\widetilde{V}_t^{\varepsilon}\cdot\nabla\theta,$$

instead of (14) and (32), because of the incompressibility of the velocity fields. Also, all the terms containing $\nabla\cdot V_t^{\varepsilon}$ and $\nabla\cdot\widetilde{V}_t^{\varepsilon}$ vanish.

The most severe term to occur in the argument for tightness (in the expression for A_4^ε) is

$$\frac{\kappa\varepsilon}{2a}\left|f'(t)\langle T_t^\varepsilon, \Delta\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle\right|$$

whose second moment can be bounded as

$$\begin{aligned}
&\frac{\kappa\varepsilon}{2a}\sqrt{\mathbb{E}\left|f'(t)\langle T_t^\varepsilon, \nabla^2\tilde{\mathscr{V}}_t^\varepsilon(\theta)\rangle\right|^2}\\
&\quad\leqslant C_1\,\frac{\kappa\varepsilon}{2a}\,\|f'\|_\infty\,\|T_0\|_\infty\left(\int_{|x|<M}\mathbb{E}[|\Delta\tilde{V}_t^\varepsilon|^2]\,dx\right)^{1/2}\\
&\quad\leqslant C_2\kappa\varepsilon\times\begin{cases}\ell_1^{\alpha+2\beta-3}, & \text{for}\quad \alpha+2\beta<3\\ \sqrt{\log(1/\ell_1)}, & \text{for}\quad \alpha+2\beta=3\\ 1, & \text{for}\quad \alpha+2\beta>3.\end{cases}
\end{aligned}\tag{52}$$

Other possibly divergent terms occurring in identifying the limit can be controlled similarly. For instance, the most severe term without the prefactor κ occurs in $R_3^\varepsilon(t)$ and can be bounded as

$$\begin{aligned}
&\varepsilon\mathbb{E}\left|\langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta)))\rangle\right|\\
&\quad\leqslant \varepsilon\,\|T_0\|_\infty\,\mathbb{E}^{1/2}\left|\mathscr{V}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\tilde{\mathscr{V}}_t^\varepsilon(\theta)))\right|^2\\
&\quad\leqslant C_1\varepsilon\,\|T_0\|_\infty\left(\int_{|x|<M}\mathbb{E}\,|V_t^\varepsilon|^2\,dx\right)^{1/2}\\
&\qquad\times\left(\int_{|x|<M}\mathbb{E}[|\tilde{V}_t^\varepsilon|^2]\,\mathbb{E}[|\nabla^2\tilde{V}_t^\varepsilon|^2]\,dx+\int_{|x|<M}\mathbb{E}[|\nabla\tilde{V}_t^\varepsilon|^4]\,dx\right)^{1/2}\\
&\quad\leqslant C_2\varepsilon\left(\int_{|x|<M}\mathbb{E}[|\nabla^2\tilde{V}_t^\varepsilon|^2]\,dx\right)^{1/2}
\end{aligned}\tag{53}$$

by the Gaussianity of the fields. The right side of (52) and (53) tends to zero if either

$$\alpha+2\beta>3$$

or

$$\alpha+2\beta=3,\qquad \lim_{\varepsilon\to0}\varepsilon\sqrt{\log(1/\ell_1)}=0\tag{54}$$

or

$$\alpha + 2\beta < 3, \qquad \lim_{\varepsilon \to 0} \varepsilon \ell_1^{\alpha+2\beta-3} = 0 \tag{55}$$

is satisfied. The term involving $\varepsilon \langle T_t^\varepsilon, \mathscr{V}_t^\varepsilon(G_\theta^{(2)} T_t^\varepsilon) \rangle$ can be similarly estimated.

The most severe term involving the pre-factor κ occurs in R_3^ε and can be bounded as

$$\kappa \varepsilon^2 \mathbb{E}\, |\langle T_t^\varepsilon, \Delta \widetilde{\mathscr{V}}_t^\varepsilon(\widetilde{\mathscr{V}}_t^\varepsilon(\theta)) \rangle| \leqslant C\kappa\varepsilon^2 \, \|T_0\|_\infty \left(\int_{|x|<M} \mathbb{E}[|\nabla^3 \widetilde{V}_t^\varepsilon|^2] \right)^{1/2}$$
$$\sim \begin{cases} \kappa\varepsilon^2, & \text{for} \quad \alpha + 2\beta > 4 \\ \kappa\varepsilon^2 \sqrt{\log(1/\ell_1)}, & \text{for} \quad \alpha + 2\beta = 4 \\ \kappa\varepsilon^2 \ell_1^{\alpha+2\beta-4}, & \text{for} \quad \alpha + 2\beta < 4 \end{cases} \tag{56}$$

the right side of which tends to zero if either

$$\alpha + 2\beta > 4$$

or

$$\alpha + 2\beta = 4, \qquad \lim_{\varepsilon \to 0} \kappa\varepsilon^2 \sqrt{\log(1/\ell_1)} = 0$$

or

$$3 < \alpha + 2\beta < 4, \qquad \lim_{\varepsilon \to 0} \kappa\varepsilon^2 \ell_1^{\alpha+2\beta-4} = 0 \tag{57}$$

or

$$2 < \alpha + 2\beta < 3, \qquad \lim_{\varepsilon \to 0} \kappa\varepsilon^2 \ell_1^{\alpha+2\beta-4} = \lim_{\varepsilon \to 0} \varepsilon \ell_1^{\alpha+2\beta-3} = 0.$$

Note that for $\alpha + 2\beta \leqslant 2$ the condition (54) or (55) implies that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ell_1^{\alpha+2\beta-4} = 0.$$

Finally we note that in the limit $(\varepsilon, \ell_1 \to 0)$ the limiting martingale is, instead of (51),

$$M_t(\theta) = f(t) - \int_0^t \left\{ f'(s) \left[\frac{\kappa_0}{2} \langle T_s, \Delta\theta \rangle + \frac{1}{a} \bar{A}_3^{(1)}(T_s) \right] + \frac{1}{a} f''(s) \bar{A}_2^{(1)}(T_s) \right\} ds \tag{58}$$

where

$$\bar{A}_2^{(1)} = \lim_{\ell_1 \to 0} A_2^{(1)}, \qquad \bar{A}_3^{(1)} = \lim_{\ell_1 \to 0} A_3^{(1)}$$

and the limiting process T_t is the (assumed unique) distributional solution to the martingale problem associated with the SDE

$$dT_t = \left(\frac{\kappa_0}{2}\Delta + \frac{1}{a}\bar{\mathscr{B}}\right) T_t\, dt + \sqrt{2}\, a^{-1/2}\, \nabla T_t \cdot d\bar{W}_t^{(1)}$$

$$= \frac{\kappa_0}{2}\Delta T_t\, dt + \sqrt{2}\, a^{-1/2}\, \nabla T_t \circ d\bar{W}_t^{(1)}$$

where $\bar{W}_t^{(1)}$ is the Brownian vector field with the spatial covariance $\bar{\Gamma}^{(1)}(x-y)$ given in (24) and the operator $\bar{\mathscr{B}}$ is given in (25).

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